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ON MIXED VARIATIONAL FORMULATIONS OF LINEAR ELASTICITY USING NONSYMMETRIC STRESSES AND DISPLACEMENTS†

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Abstract—Enforcement of the symmetry constraint on the stress tensor in dual mixed variational principles using nonsymmetric stresses is usually achieved through introducing the rotations as Lagrange multipliers. This paper presents a new way for imposition of the symmetry requirement on the stress tensor assumed to be not *a priori* symmetric. The central idea in our approach is to find symmetry-equivalent conditions that can be incorporated into the corresponding variational principles using the displacements as Lagrange multipliers. © 1997 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The easiest way to ensure the symmetry of a second-order tensor is to consider only its six appropriate components to be independent. This fact is consistently utilized in the classical dual and dual mixed variational formulations of linear elasticity, when the stress tensor is assumed to be *a priori* symmetric (Oden and Reddy, 1983; Washizu, 1975). Using six stress variables instead of nine means not only the fulfillment of the three rotational equilibrium equations, but also seems to be advantageous from the point of view of numerical approximation. However, the rather simple condition of symmetry, when satisfied *a priori*, is the indirect source of many computational difficulties experienced in dual and dual mixed finite element formulations.

The symmetry constraint is responsible for the fact, for example, that the dual equation system of elasticity is a fourth-order (elliptic) problem. Consequently, a pure equilibrium formulation, based on the principle of minimum complementary energy, requires C^1 continuous approximation of the independent stress components or, equivalently, of the three non-zero components of the second-order stress function tensor (assume that the variables are approximated conformly). Efforts made by Fraeijs de Veubeke and his school to construct effective and stable equilibrium elements for structural problems have led to the recognition that many of the difficulties involved in stress-based finite element formulations can be avoided, if the stress tensor is not assumed to be *a priori* symmetric (Fraeijs de Veubeke, 1975). A weaker imposition of the constraint of symmetry can be achieved by introducing the rotations as Lagrange multipliers to enforce the rotational equilibrium equations into the dual mixed variational principle of Fraeijs de Veubeke (1972). Equilibrium elements based on this principle require only C^0 continuous approximations of the independent stresses or first-order stress functions and have been developed, for instance, by Fraeijs de Veubeke (1973; 1975), Fraeijs de Veubeke and Millard (1976), and Amara and Thomas (1979) for two-dimensional problems, and Atluri (1983), Murakawa and Atluri (1978; 1979), and Puch and Atluri (1986) for nonlinear problems.

Construction of stable and efficient mixed elements based on the Hellinger–Reissner variational principle (with symmetric stresses) has also proved to be difficult, even for the two-dimensional case. One of the possibilities to overcome the difficulties is to give up the *a priori* symmetry of the stress tensor and to impose it in a weaker sense through the introduction of the rotations as Lagrange multipliers (for other possibilities see Brezzi

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and Fortin, 1991). The resulting variational principle in terms of nonsymmetric stresses, displacements and rotations is not less than that proposed by Reissner (1965). Mixed elements based on this three-field principle have been developed by Arnold *et al.* (1984), Morley (1989) and Stenberg (1988), implementation aspects and numerical results for plane elasticity problems have been presented by Stein and Rolfes (1990) and Klaas *et al.* (1995). Although the low-order elements developed in the above mentioned papers for plane elasticity problems proved to be efficient, a possible objection to this three-field variational formulation may be that the discretization of a general three-dimensional problem requires the approximation of 15 independent variables. Arnold and Falk (1988) have overcome this problem by developing a new two-field mixed formulation in terms of nonsymmetric "pseudostresses" and displacements. Their functional is very similar to that of Hellinger–Reissner, and the true (symmetric) stresses can be easily recovered from the pseudostress tensor by linear combinations of its components. Using this principle, the only difficulty is the imposition of the prescribed tractions on the pseudostresses.

In this paper it will be shown that enforcement of the symmetry constraint on the stress tensor by another way than using rotations as independent variables is possible. The key lies in finding symmetry-equivalent conditions for the stress tensor or, more precisely, conditions that assure the vanishing of the skew-symmetric part of the nonsymmetric stress tensor. The symmetry-equivalent conditions proposed in Section 3 can be incorporated into dual mixed variational principles using the displacements as Lagrange multipliers, and the introduction of the rotations as additional multipliers is not needed. This new approach leads to a generalization of the Hellinger–Reissner variational principle, and to a new interpretation of Fraeijs de Veubeke's principle.

2. NOTATION AND PRELIMINARIES

Consider the region Ω, occupied by the elastic body, in the three-dimensional space. Let Ω be bounded by a sufficiently smooth boundary $\partial\Omega = \Gamma = \Gamma_t \cup \Gamma_u$, where Γ_t and Γ_u are disjoint closed subsets of Γ with outward unit normal $\mathbf{n}$. The elastic body is subjected to given body forces with density $\mathbf{f}$ in Ω, prescribed surface tractions $\tilde{\mathbf{t}}$ on Γ_t and prescribed displacements $\tilde{\mathbf{u}}$ on Γ_u.

The classical dual variational formulation of the linear elasticity problem is based on the principle of minimum complementary energy. Its functional,

$$\Pi_c(\mathbf{T}) = \int_{\Omega} \tfrac{1}{2} T^{kl} e_{lk} \, d\Omega - \int_{\Gamma_u} n_k T^{kl} \tilde{u}_l \, d\Gamma, \tag{1}$$

is considered over the space of all stress fields $\mathbf{T}$ satisfying the translational and rotational equilibrium equations

$$T^{kl}_{;k} + f^l = 0 \quad \text{in } \Omega, \tag{2}$$

and

$$T^{kl} - T^{lk} = 0 \quad \text{in } \Omega, \tag{3}$$

respectively, and the stress boundary conditions

$$n_k T^{kl} = \tilde{t}^l \quad \text{on } \Gamma_t. \tag{4}$$

The (symmetric) strain tensor e_{kl} is determined by the inverse stress–strain relations

$$e_{kl} = A_{klpq} T^{pq} \quad \text{in } \Omega, \tag{5}$$

where the fourth-order tensor $A_{klpq} = A_{pqkl} = A_{lkpq}$ is the elastic compliance tensor. Note that after satisfaction of the six constraints of eqns (2) and (3), the functional (1) depends only on three independent stress components, or, if second-order stress functions are used, on the three non-zero components of the second-order stress function tensor.

The advantage of equilibrium finite element formulations based on the above principle is that the variable of primary interest, the stress field, appears as a directly approximated and computed variable (in contrast to the displacement-based virtual work or potential energy approaches). It is very difficult, however, to create finite element spaces for general problems that satisfy, *a priori*, all the necessary constraints (2)–(4), and at the same time, the traction reciprocity conditions at the element boundaries. Conforming equilibrium elements for the complementary energy principle require moreover C^1 continuous approximations of the independent stress variables or, equivalently, of the independent second-order stress functions.

The origin of numerical difficulties mentioned above can be easily traced back to the symmetry requirement on the stress tensor. If the stresses are not assumed to be *a priori* symmetric, satisfaction of the translational equilibrium alone requires only first-order stress functions (Fraeijs de Veubeke, 1975). Equilibrium elements with relaxed symmetry of the stress tensor have many advantageous properties (comparing to the conventional equilibrium elements) and are based on the dual mixed variational principle proposed by Fraeijs de Veubeke (1972), originally for finite deformation problems. In linear elasticity, this principle is considered over the space of all nonsymmetric stress fields $\mathbf{T}$ satisfying the translational equilibrium equations (2) and the stress boundary conditions (4), and all skew-symmetric rotation fields $\boldsymbol{\phi}$. It seeks a saddle point of the functional

$$\mathscr{F}(\mathbf{T},\boldsymbol{\phi}) = \int_\Omega (\tfrac{1}{2}T^{kl}e_{lk} + T^{kl}\phi_{lk})\mathrm{d}\Omega - \int_{\Gamma_u} n_k T^{kl}\tilde{u}_l\,\mathrm{d}\Gamma, \tag{6}$$

where the three non-zero components of the infinitesimal rigid body rotation tensor ϕ_{lk} are the Lagrange multipliers enforcing the symmetry condition (3) into the principle. Note that after satisfying the three translational equilibrium constraints (2), this functional depends only on six independent (nonsymmetric) stress components or, equivalently, on six non-zero components of the tensor of first-order stress functions (Bertóti, 1994).

This principle allows us to construct isoparametric equilibrium elements with strongly diffusive surface tractions at the element boundaries and with independent approximations of the rotations. It should be mentioned, however, that for situations where body forces exist, it is necessary to construct particular solutions to the translational equilibrium equations. Additional implementation difficulties may arise moreover in the imposition of the prescribed surface tractions, and the introduction of the displacements (at least on the boundary Γ) is almost inevitable for general problems.

Another possibility of resolving the difficulties involved in pure equilibrium formulations is to enforce the translational equilibrium constraint (2) into a Hellinger–Reissner type variational principle. The functional of this principle,

$$\mathscr{H}(\mathbf{T},\mathbf{u}) = \int_\Omega (\tfrac{1}{2}T^{kl}e_{lk} + T^{kl}_{;k}u_l + f^l u_l)\,\mathrm{d}\Omega - \int_{\Gamma_u} n_k T^{kl}\tilde{u}_l\,\mathrm{d}\Gamma, \tag{7}$$

is considered over the space of all symmetric stress fields $\mathbf{T}$ satisfying the stress boundary conditions (4) and all displacement fields $\mathbf{u}$ (the Lagrange multiplier). It turned out, however, to be difficult to build stable finite element spaces for this principle, even for plane elasticity problems. The basic source of difficulties, as discussed by several authors (see Arnold and Falk, 1988; Brezzi and Fortin, 1991; Morley, 1989; Stenberg, 1988, for example), is that the stress tensor in the Hellinger–Reissner principle is symmetric (and this condition does not allow the direct use of those stable spaces developed for scalar second-order problems).

One way to overcome the aforementioned problem is the modification of the Hellinger–Reissner principle by introducing the rotations, again, as independent variables (see also

Brezzi and Fortin, 1991). The resulting three-field dual mixed variational principle, which was proposed by Reissner (1965), can be written in the following form:

$$\mathscr{R}(\mathbf{T},\mathbf{u},\boldsymbol{\phi}) = \int_{\Omega} (\tfrac{1}{2} T^{kl} e_{lk} + T^{kl}_{;k} u_l + f^l u_l + T^{kl}\phi_{lk})\,\mathrm{d}\Omega - \int_{\Gamma_u} n_k T^{kl} \tilde{u}_l \,\mathrm{d}\Gamma. \tag{8}$$

Here, both $\mathbf{u}$ and $\boldsymbol{\phi}$ are Lagrange multipliers to enforce the translational as well as the rotational equilibrium eqns (2) and (3), respectively, into the principle. The mixed finite elements developed by Arnold *et al.* (1984), Morley (1989) and Stenberg (1988) are based essentially on this variational principle. Reissner's principle was also considered by Hughes and Brezzi (1989), where construction of variational formulations with drilling degrees of freedom was investigated.

Enforcement of the symmetry constraint on the stress tensor through the introduction of the rotations as independent variables is quite a "natural" way, as the term $T^{kl}\phi_{lk}$ in the above principles immediately implies the three rotational equilibrium equations (3). However, it is a hard task to build numerically efficient (and stable) finite elements for three-dimensional problems using a three-field mixed variational principle with 15 independent variables. A resolution for this problem was given by Arnold and Falk (1988) by the development of a new two-field mixed variational formulation in terms of nonsymmetric pseudostresses and displacements.

In the next sections, it will be shown how two-field mixed variational principles with nonsymmetric stresses and displacements can be derived from Reissner's as well as Fraeijs de Veubeke's variational principle, finding symmetry equivalent conditions for the stress tensor. We assume from now, for simplicity, that Ω is a simply connected domain.

3. SYMMETRY-EQUIVALENT CONDITIONS FOR THE STRESS TENSOR

Consider the additive decomposition of the nonsymmetric stress tensor $\mathbf{T}$:

$$\mathbf{T} = \boldsymbol{\sigma} + \boldsymbol{\tau}, \tag{9}$$

where

$$\boldsymbol{\sigma} := \operatorname{symm}\mathbf{T} = \tfrac{1}{2}(\mathbf{T}+\mathbf{T}^t) \tag{10}$$

and

$$\boldsymbol{\tau} := \operatorname{skew}\mathbf{T} = \tfrac{1}{2}(\mathbf{T}-\mathbf{T}^t) \tag{11}$$

are, respectively, the symmetric and skew-symmetric parts of $\mathbf{T}$. Making use of eqn (9), the translational equilibrium equations (2) and the stress boundary conditions (4) can be written in the following forms:

$$\sigma^{kl}_{;k} + \tau^{kl}_{;k} + f^l = 0 \quad \text{in } \Omega, \tag{12}$$

and

$$n_k \sigma^{kl} + n_k \tau^{kl} = \tilde{t}_l \quad \text{on } \Gamma_t. \tag{13}$$

If $\mathbf{T}$ is symmetric, it coincides with $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ vanishes. In this case, it is obvious that $\tau^{kl}_{;k} = 0$ in Ω and $n_k\tau^{kl} = 0$ on Γ_t and eqns (12) and (13) hold for the symmetric stress tensor $\boldsymbol{\sigma}$. Conversely, if $\mathbf{T}$ is nonsymmetric but $\boldsymbol{\tau}$ satisfies the conditions $\tau^{kl}_{;k} = 0$ in Ω and $n_k\tau^{kl} = 0$ on Γ_t, eqns (12) and (13) hold again for the symmetric part of the stress tensor $\boldsymbol{\sigma}$. Do these two conditions imply the vanishing of $\boldsymbol{\tau}$ in Ω and, thus, the symmetry of $\mathbf{T}$? By the following Lemma and its proof it will be shown that if $n_k\tau^{kl} = 0$ holds also on Γ_u, then the above two conditions assure the symmetry of the stress tensor $\mathbf{T}$ in Ω.

Lemma 1

Let τ be divergence-free in Ω and let its traction be zero on the whole boundary $\Gamma = \partial\Omega$, i.e.

$$\tau^{kl}_{;k} = 0 \quad \text{in } \Omega, \tag{14}$$

and

$$n_k \tau^{kl} = 0 \quad \text{on } \Gamma = \partial\Omega. \tag{15}$$

Then $\tau^{kl} = 0$ in Ω.

Proof

Consider an arbitrary vector field, v_l, in Ω. Let eqn (14) be multiplied by v_l and then integrated over the domain Ω. We obtain:

$$\int_\Omega \tau^{kl}_{;k} v_l \, \mathrm{d}\Omega = 0. \tag{16}$$

Applying the divergence theorem, eqn (16) can be transformed into

$$\int_\Gamma n_k \tau^{kl} v_l \, \mathrm{d}\Gamma - \int_\Omega \tau^{kl} v_{l;k} \, \mathrm{d}\Omega = 0, \tag{17}$$

and since eqn (15) holds, we have

$$\int_\Omega \tau^{kl} v_{l;k} \, \mathrm{d}\Omega = 0. \tag{18}$$

An important observation is that vanishing of τ does not follow immediately from eqn (18), as all the nine components of the gradient tensor $v_{l;k}$ cannot be chosen arbitrarily; three components of $v_{l;k}$ always determine the vector v_l and the other six non-independent gradient-components. In a general curvilinear coordinate system, there exist many possible choices for these three arbitrary components of $v_{l;k}$. Considering the fact, however, that our choice should be independent of the coordinate system in which we are working (think of the most important Cartesian one), the three arbitrary components must stand in three different rows of the matrix of $v_{l;k}$. Of course, eqn (18) must hold for every possible choice of the arbitrary triads (they are $3^3 = 27$ in number) that gives the solution for the vector field v_l.

Taking into account the skew-symmetry of τ^{kl}, eqn (18) takes the form

$$\int_\Omega [\tau^{12}(v_{2;1} - v_{1;2}) + \tau^{23}(v_{3;2} - v_{2;3}) + \tau^{31}(v_{1;3} - v_{3;1})] \, \mathrm{d}\Omega = 0. \tag{19}$$

According to the above considerations, eqn (19) must also hold when either of the triads $v_{2;1}$, $v_{3;2}$, $v_{1;3}$, or $v_{3;1}$, $v_{2;3}$, $v_{1;2}$ is chosen. If the first triad is arbitrary, the second one is determined, and vice versa, but the three terms in parenthesis in eqn (19) remain arbitrary in either of the cases. Since the integral (19) is independent of Ω, this means that the three non-zero components of τ^{kl} must vanish, i.e. $\tau = 0$ in Ω. □

The consequence of Lemma 1 is that eqns (14) and (15) are equivalent to the symmetry of the stress tensor $\mathbf{T}$ in Ω. This fact makes it possible to enforce the symmetry requirement on the stress tensor by a different manner than it was done in Section 2, i.e. using the displacements as Lagrange multipliers.

4. A GENERALIZATION OF HELLINGER–REISSNER PRINCIPLE IN TERMS OF NONSYMMETRIC STRESSES AND DISPLACEMENTS

The basic idea, which leads to a new interpretation of the three-field Reissner's principle with functional (8), is that the rotation tensor be considered no more as an independent variable, but as the skew-symmetric part of the displacement gradient tensor. In view of the previous section and using the identity

$$\int_\Gamma n_k \tau^{kl} u_l \,\mathrm{d}\Gamma - \int_\Omega \tau^{kl}_{;k} u_l \,\mathrm{d}\Omega = \int_\Omega \tau^{kl} u_{l;k} \,\mathrm{d}\Omega = \int_\Omega T^{kl}\phi_{lk}\,\mathrm{d}\Omega, \tag{20}$$

the term $T^{kl}\phi_{lk}$ in eqn (8) can be replaced by $\tau^{kl}u_{l;k}$, where $u_{l;k}$ depends on the three displacement components u_l. The resulting two-field dual mixed variational principle in terms of nonsymmetric stresses and displacements with functional

$$\mathscr{H}^*(\mathbf{T},\mathbf{u}) = \int_\Omega [\tfrac{1}{2}T^{kl}e_{lk} + T^{kl}_{;k}u_l + f^l u_l + \tau^{kl}u_{l;k}]\,\mathrm{d}\Omega - \int_{\Gamma_u} n_k T^{kl}\tilde{u}_l \,\mathrm{d}\Gamma, \tag{21}$$

and with subsidiary conditions

$$T^{kl} = \sigma^{kl} + \tau^{kl} \quad \text{in } \Omega, \tag{22}$$

$$\tau^{kl}_{;k} = 0 \quad \text{in } \Omega, \tag{23}$$

$$e_{kl} = A_{klpq}T^{pq} \quad \text{in } \Omega, \tag{24}$$

and

$$n_k T^{kl} = \tilde{t}^l \quad \text{on } \Gamma_t \tag{25}$$

can be considered as a generalization of the Hellinger–Reissner variational principle.
The stationary value of $\mathscr{H}^*(\mathbf{T},\mathbf{u})$ is found by setting its first variation equal to zero. Applying the divergence theorem we can write:

$$\delta\mathscr{H}^*(\delta\mathbf{T},\delta\mathbf{u}) = \int_\Omega [\delta T^{kl}(e_{lk} - u_{l;k}) + \delta u_l(T^{kl}_{;k} + f^l) + \delta\tau^{kl}u_{l;k}]\,\mathrm{d}\Omega$$
$$+ \int_\Gamma (n_k\delta T^{kl}u_l + n_k\tau^{kl}\delta u_l)\,\mathrm{d}\Gamma - \int_{\Gamma_u} n_k\delta T^{kl}\tilde{u}_l\,\mathrm{d}\Gamma = 0. \tag{26}$$

It is important to see that $\delta\mathrm{T}$ is constrained not only on the surface part Γ_t, because of eqn (25), but also in Ω, as its skew-symmetric part, $\delta\boldsymbol{\tau}$, is constrained through eqn (23) in Ω. Thus, only its symmetric part, $\delta\boldsymbol{\sigma}$, has arbitrary variation in Ω. However, by the use of the additive decomposition of the displacement gradient tensor

$$u_{l;k} = u_{(l;k)} + u_{[l;k]}, \tag{27}$$

where $u_{(l;k)}$ and $u_{[l;k]}$ are the symmetric and skew-symmetric parts of $u_{l;k}$, respectively, and the identity

$$\delta\tau^{kl}u_{l;k} = \tfrac{1}{2}(\delta T^{kl} - \delta T^{lk})u_{l;k} = \delta T^{kl}\tfrac{1}{2}(u_{l;k} - u_{k;l}) = \delta T^{kl}u_{[l;k]}, \tag{28}$$

eqn (26) can be transformed into

$$\delta\mathscr{H}^*(\delta\mathbf{T},\delta\mathbf{u}) = \int_\Omega [\delta\sigma^{kl}(e_{lk}-u_{(l;k)})+\delta u_l(T^{kl}_{;k}+f^l)]\,d\Omega + \int_\Gamma n_k\tau^{kl}\delta u_l\,d\Gamma + \int_{\Gamma_u} n_k\delta T^{kl}(u_l-\tilde{u}_l)\,d\Gamma = 0, \quad (29)$$

and we can observe that the coefficient of $\delta\tau^{kl}$ has vanished, i.e. free variation of $\delta\tau^{kl}$ is not needed at all in Ω.

Variational equation (29) enforces the Euler equations and natural boundary conditions of the generalized Hellinger–Reissner principle, viz., the six strain–displacement relations

$$e_{lk}-u_{(l;k)} = 0 \quad \text{in } \Omega, \quad (30)$$

three translational equilibrium equations

$$T^{kl}_{;k}+f^l = 0 \quad \text{in } \Omega, \quad (31)$$

three zero-traction conditions for the skew-symmetric stress tensor

$$n_k\tau^{kl} = 0 \quad \text{on } \Gamma, \quad (32)$$

and three displacement boundary conditions

$$u_l-\tilde{u}_l = 0 \quad \text{on } \Gamma_u. \quad (33)$$

Note that, following from the results of Section 3, eqns (32) together with (23) ensure the symmetry of the stress tensor $\mathbf{T}$ (the vanishing of $\boldsymbol{\tau}$).

Remark 4.1.

In the above principle, the nonsymmetry of the stress tensor, $\mathbf{T}$, is constrained by the fact that its skew-symmetric part, $\boldsymbol{\tau}$, must be divergence free in Ω, according to eqn (23). In practical applications, therefore, it is advisable to consider $\mathbf{T}$ as the sum of an arbitrary symmetric tensor and a skew-symmetric tensor that satisfy eqn (23). This latter condition can be easily and identically satisfied, if $\boldsymbol{\tau}$ is derived from an arbitrary scalar function, χ, as

$$\tau^{kl} = \varepsilon^{klm}\chi_{;m} \quad \text{in } \Omega,$$

where ε^{klm} is the contravariant permutation tensor. Then,

$$T^{kl} = \sigma^{kl}+\varepsilon^{klm}\chi_{;m},$$

which means, that the generalized Hellinger–Reissner principle has 10 independent variables: six symmetric stress components σ^{kl}, one arbitrary scalar function χ and three displacement components u_l. In finite element procedures, imposition of the stress boundary conditions for $\mathbf{T}$ can be achieved, for example, by applying the λ-multiplier technique (Brezzi and Fortin, 1991).

5. FRAEIJS DE VEUBEKE'S PRINCIPLE IN TERMS OF NONSYMMETRIC STRESSES AND DISPLACEMENTS

In the original variational principle of Fraeijs de Veubeke with functional (6), the displacements do not appear as variables; they can be obtained from the strains and rotations by integration. According to the considerations made at the beginning of Section 4, this two-field principle can be modified by replacing the term $\mathbf{T}^{kl}\phi_{lk}$ by $\tau^{kl}u_{l;k}$. The result is a two-field dual mixed variational principle in terms of nonsymmetric stresses and displacements with functional

$$\mathscr{F}^*(\mathbf{T},\mathbf{u}) = \int_\Omega (T^{kl}e_{lk}+\tau^{kl}u_{l;k})\,\mathrm{d}\Omega - \int_{\Gamma_u} n_k T^{kl}\tilde{u}_l\,\mathrm{d}\Gamma, \tag{34}$$

and with subsidiary conditions

$$T^{kl}_{;k}+f^l = 0 \quad \text{in } \Omega, \tag{35}$$

$$e_{kl} = A_{klpq}T^{pq} \quad \text{in } \Omega, \tag{36}$$

$$\tau^{kl} = \tfrac{1}{2}(T^{kl}-T^{lk}) \quad \text{in } \Omega, \tag{37}$$

and

$$n_k T^{kl} = \tilde{t}^l \quad \text{on } \Gamma_t. \tag{38}$$

Derivation of the independent Euler equations and natural boundary conditions of this principle is very similar to that presented by Bertóti (1994), where, instead of the displacements, the rotations as independent variables were used. A brief derivation for a simply connected domain with two surface parts Γ_u and Γ_t ($\Gamma_u \cup \Gamma_t = \Gamma = \partial\Omega$) is given in the following. The boundary curve between Γ_u and Γ_t is denoted by g.

The translational equilibrium eqns (35) can be identically satisfied by introducing the nonsymmetric tensor of first-order stress functions $\psi_{\underline{r}}^{\cdot l}$, such that

$$T^{kl} = \varepsilon^{km\underline{r}}\psi_{\underline{r};m}^{\cdot l} + \hat{T}^{kl}, \tag{39}$$

where ε^{kmr} is the contravariant permutation tensor and $\hat{T}^{kl}$ is a particular solution to eqn (35). The underlined index $\underline{r}$ indicates that, following from the indeterminacy property of the first-order stress functions, $\psi_r^{\cdot l}$ has only six independent non-zero components, $\psi_{\underline{r}}^{\cdot l}$. The three zero-valued components can always be chosen in such a way, independently of the coordinate system in which we are working, that they stand in three different columns of the matrix of $\psi_r^{\cdot l}$ (Bertóti, 1994).

The stationary value of eqn (34) can be obtained by setting its first variation equal to zero. During the course of integral transformations, one has to take into account that

(i) following from eqn (39), equilibrated stress variations can be expressed by first-order stress function variations $\delta\psi_{\underline{r}}^{\cdot l}$ as

$$\delta T^{kl} = \varepsilon^{km\underline{r}}\,\delta\psi_{\underline{r};m}^{\cdot l}; \tag{40}$$

(ii) stress boundary conditions (38) in terms of $\delta\psi_{\underline{r}}^{\cdot l}$ take the form

$$n_k\delta T^{kl} = n_k\varepsilon^{km\underline{r}}\delta\psi_{\underline{r};m}^{\cdot l} = 0 \quad \text{on } \Gamma_t, \tag{41}$$

and this equation can be identically satisfied by introducing an arbitrary vector field v^l defined on the boundary part Γ_t as

$$\delta\psi_{\underline{r}}^{\cdot l} = v^l_{;\underline{r}} \quad \text{on } \Gamma_t, \tag{42}$$

provided equation

$$\breve{t}^r(\delta\psi^{\cdot l}_{\underline{r}} - v^l_{;\underline{r}}) = 0 \quad \text{on } g \tag{43}$$

holds (g is the boundary curve between the surface parts Γ_u and Γ_t), where $\breve{\mathbf{t}}$ is the unit tangent to g (for proof see Bertóti, 1994).

After integral transformations, taking into account eqns (40)–(43) and applying the Gauss as well as the Stokes theorems, and also the identity (28), the stationary value of eqn (34) is expressed by the variational equation

$$\delta\mathscr{F}^*(\delta\psi^{\cdot l}_{\underline{r}}, \delta u_l) = \int_\Omega \varepsilon^{km\underline{r}}(e_{lk} + u_{[l;k]})_{;m}\delta\psi^{\cdot l}_{\underline{r}}\,\mathrm{d}\Omega$$
$$- \int_\Omega \tau^{kl}_{;k}\delta u_l\,\mathrm{d}\Omega + \int_\Gamma n_k\tau^{kl}\delta u_l\,\mathrm{d}\Gamma + \int_{\Gamma_u} n_m\varepsilon^{km\underline{r}}(e_{lk} + u_{[l;k]} - \tilde{u}_{l;k})\,\delta\psi^{\cdot l}_{\underline{r}}\,\mathrm{d}\Gamma$$
$$- \int_{\Gamma_t} n_m\varepsilon^{kmr}(e_{lk} + u_{[l;k]})_{;r}v^l\,\mathrm{d}\Gamma + \oint_g \breve{t}^k(e_{lk} + u_{[l;k]} - \tilde{u}_{l;k})v^l\,\mathrm{d}s = 0, \tag{44}$$

with six arbitrary first-order stress function variations $\delta\psi^{\cdot l}_{\underline{r}}$ in Ω and on the boundary part Γ_u, with three arbitrary displacement variations δ_{u_l} in Ω and on Γ, and with an arbitrary vector field v^l on Γ_t and g.

Variational eqn (44) implies the independent Euler equations and natural boundary conditions of the principle under consideration, i.e. the six independent first-order compatibility equations

$$\varepsilon^{km\underline{r}}(e_{lk} + u_{[l;k]})_{;m} = 0 \quad \text{in } \Omega, \tag{45}$$

the symmetry-equivalent conditions

$$\tau^{kl}_{;k} = 0 \quad \text{in } \Omega \tag{46}$$

and

$$n_k\tau^{kl} = 0 \quad \text{on } \Gamma, \tag{47}$$

three first-order compatibility boundary conditions

$$n_m\varepsilon^{kmr}(e_{lk} + u_{[l;k]})_{;r} = 0 \quad \text{on } \Gamma_t, \tag{48}$$

six independent displacement boundary conditions

$$n_m\varepsilon^{km\underline{r}}(e_{lk} + u_{[l;k]} - \tilde{u}_{l;k}) = 0 \quad \text{on } \Gamma_u, \tag{49}$$

and three continuity conditions for the displacement gradient between the two surface parts Γ_u and Γ_t

$$\breve{t}^k(e_{lk} + u_{[l;k]} - \tilde{u}_{l;k}) = 0 \quad \text{on } g. \tag{50}$$

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